

A Remark on the Stability of Peakons for the Degasperis-Procesi Equation

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Abstract

In this paper, we present a new argument (see Lemma 3.4) that allows us to simplify the proof of stability of peakons established in Lin and Liu (2009) (Theorem 1.1).

1 Introduction

In this paper, we consider the Degasperis-Procesi equation (DP)

$$u_t - u_{txx} + 4uu_x = 3u_xu_{xx} + uu_{xxx}, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \quad (1.1)$$

with $u(0) = u_0 \in L^2(\mathbb{R})$ and $(1 - \partial_x^2)u_0 \in \mathcal{M}^+(\mathbb{R})$.

The DP equation is completely integrable (see [3]) and has been proved to be physically relevant for water waves (see [1]). It possesses, among others, the following conservation laws

$$E(u) = \int_{\mathbb{R}} yv = \int_{\mathbb{R}} (4v^2 + 5v_x^2 + v_{xx}^2), \quad F(u) = \int_{\mathbb{R}} u^3 = \int_{\mathbb{R}} (-v_{xx}^3 + 12vv_{xx}^2 - 48v^2v_{xx} + 64v^3), \quad (1.2)$$

where $y = (1 - \partial_x^2)u$ and $v = (4 - \partial_x^2)^{-1}u$. One can notice that the conservation law $E(\cdot)$ is equivalent to $\|\cdot\|_{L^2(\mathbb{R})}^2$. Indeed, using integration by parts (we assume that $u(\pm\infty) = v(\pm\infty) = v_x(\pm\infty) = 0$), it holds

$$\|u\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} u^2 = \int_{\mathbb{R}} (4v - v_{xx})^2 = \int_{\mathbb{R}} (16v^2 + 8v_x^2 + v_{xx}^2) \sim E(u). \quad (1.3)$$

In the sequel we will denote

$$\|u\|_{\mathcal{H}} = \sqrt{E(u)}. \quad (1.4)$$

Applying $(1 - \partial_x^2)^{-1}(\cdot)$ to (1.1), we obtain

$$u_t + \frac{1}{2}\partial_x u^2 + \frac{3}{2}(1 - \partial_x^2)^{-1}\partial_x u^2 = 0, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}. \quad (1.5)$$

In this form, the DP equation admits explicit solitary waves called *peakons* (see [3]) that are defined by

$$u(t, x) = \varphi_c(x - ct) = c\varphi(x - ct) = ce^{-|x-ct|}, \quad c \in \mathbb{R}^*, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}. \quad (1.6)$$

Our goal is to simplify the proof given in [7] of the stability of a single peakon for the DP equation. Recall that the proof of the stability for the Camassa-Holm equation (CH) in [2] follows from two integral relations between two conservation laws of CH, $\max_{\mathbb{R}} u$ and functions related to u . In [7] the proof is more complicated, since all the local maxima and minima of $v = (4 - \partial_x^2)^{-1}u$ are involved in the relations. In this paper we present a simplification of this proof, where only the maximum of v is involved in the relations. Our proof is thus closer to the proof for CH in [2]. The main idea is the following: since u is L^2 -close to the peakon $\varphi_c(\cdot - \xi)$, for some $\xi \in \mathbb{R}$, and $(1 - \partial_x^2)u \in \mathcal{M}^+(\mathbb{R})$, it is easy to check that u is actually C^0 -close to the peakon, and thus v is C^2 -close to the *smooth-peakon*:

$$\rho_c(x - \xi) = (4 - \partial_x^2)^{-1}\varphi_c(x - \xi) = \frac{c}{3}e^{-|x-\xi|} - \frac{c}{6}e^{-2|x-\xi|}, \quad x \in \mathbb{R}. \quad (1.7)$$

First, since ρ_c , ρ'_c and ρ''_c are very small with respect to the amplitude c outside of the interval $\Theta_0 = [-6.7, 6.7]$, we can restrict ourself to study v on $\Theta_\xi = [\xi - 6.7, \xi + 6.7]$. Now we observe that ρ''_c has strictly negative values in the interval $\mathcal{V}_0 = [-\ln\sqrt{2}, \ln\sqrt{2}]$, with ρ'_c strictly positive on $[-6.7, -\ln\sqrt{2}]$ and ρ'_c strictly negative on $[\ln\sqrt{2}, 6.7]$. This forces v_x to change sign only one time on Θ_ξ , and thus v has only one local extremum (which is a maximum) on Θ_ξ . This fact will considerably simplify the proof of the stability.

2 Preliminaries

In this section, we briefly recall the global well-posedness results for the Cauchy problem of the DP equation (see [5] and [8]), and its consequences. For I a finite or infinite time interval of \mathbb{R}_+ , we denote by $\mathcal{X}(I)$ the function space ¹

$$\mathcal{X}(I) = \{u \in C(I; H^1(\mathbb{R})) \cap L^\infty(I; W^{1,1}(\mathbb{R})), \quad u_x \in L^\infty(I; BV(\mathbb{R}))\}. \quad (2.1)$$

Theorem 2.1 (Global Weak Solution; See [5] and [8]). *Assume that $u_0 \in L^2(\mathbb{R})$ with $y_0 = (1 - \partial_x^2)u_0 \in \mathcal{M}^+(\mathbb{R})$. Then the DP equation has a unique global weak solution $u \in \mathcal{X}(\mathbb{R}_+)$ such that*

$$y(t, \cdot) = (1 - \partial_x^2)u(t, \cdot) \in \mathcal{M}^+(\mathbb{R}), \quad \forall t \in \mathbb{R}_+ \quad (2.2)$$

and

$$|u_x(t, x)| \leq u(t, x), \quad \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}. \quad (2.3)$$

Moreover $E(\cdot)$ and $F(\cdot)$ are conserved by the flow.

Remark 2.1 (Control of L^∞ Norm by L^2 Norm). (2.3) and the well-known Sobolev embedding of $H^1(\mathbb{R})$ into $L^\infty(\mathbb{R})$ lead to

$$\|u\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2}}\|u\|_{H^1(\mathbb{R})} \leq \|u\|_{L^2(\mathbb{R})}. \quad (2.4)$$

3 Stability of peakons

In this section, we present our simplification of the proof of stability of peakons for the DP equation.

Theorem 3.1 (Stability of Peakons). *Let $u \in \mathcal{X}([0, T])$, with $0 < T \leq +\infty$, be a solution of the DP equation and φ_c be the peakon defined in (1.6), traveling to the right at the speed $c > 0$. There exist $C > 0$ and $\varepsilon_0 > 0$ only depending on the speed c , such that if*

$$y_0 = (1 - \partial_x^2)u_0 \in \mathcal{M}^+(\mathbb{R}) \quad (3.1)$$

¹ $W^{1,1}(\mathbb{R})$ is the space of $L^1(\mathbb{R})$ functions with derivatives in $L^1(\mathbb{R})$ and $BV(\mathbb{R})$ is the space of function with bounded variation.

and

$$\|u_0 - \varphi_c\|_{\mathcal{H}} \leq \varepsilon^2, \quad \text{with } 0 < \varepsilon < \varepsilon_0, \quad (3.2)$$

then

$$\|u(t, \cdot) - \varphi_c(\cdot - \xi(t))\|_{\mathcal{H}} \leq C\sqrt{\varepsilon}, \quad \forall t \in [0, T[, \quad (3.3)$$

where $\xi(t) \in \mathbb{R}$ is the only point where the function $v(t, \cdot) = (4 - \partial_x^2)^{-1}u(t, \cdot)$ attains its maximum.

We first recall that $E(u) \sim E(\varphi_c)$ and $F(u) \sim F(\varphi_c)$ in \mathbb{R} , if $u \sim \varphi_c$ in $L^2(\mathbb{R})$, with $y \in \mathcal{M}^+(\mathbb{R})$ (see for instance [7] or [6]).

Lemma 3.1 (Control of Distances Between Energies; See [6]). *Let $u \in L^2(\mathbb{R})$ with $y = (1 - \partial_x^2)u \in \mathcal{M}^+(\mathbb{R})$. If $\|u - \varphi_c\|_{\mathcal{H}} \leq \varepsilon^2$, then*

$$|E(u) - E(\varphi_c)| \leq O(\varepsilon^2) \quad (3.4)$$

and

$$|F(u) - F(\varphi_c)| \leq O(\varepsilon^2), \quad (3.5)$$

where $O(\cdot)$ only depends on the speed c .

To prove Theorem 3.1, by the conservation of $E(\cdot)$, $F(\cdot)$ and the continuity of the map $t \mapsto u(t)$ from $[0, T[$ to \mathcal{H} (since $\mathcal{H} \simeq L^2$), it suffices to prove that for any function $u \in L^2(\mathbb{R})$ satisfying $y = (1 - \partial_x^2)u \in \mathcal{M}^+(\mathbb{R})$, (3.4) and (3.5), if

$$\inf_{z \in \mathbb{R}} \|u - \varphi_c(\cdot - z)\|_{\mathcal{H}} \leq \varepsilon^{1/4}, \quad (3.6)$$

then

$$\|u - \varphi_c(\cdot - \xi)\|_{\mathcal{H}} \leq C\sqrt{\varepsilon}, \quad (3.7)$$

where $\xi \in \mathbb{R}$ is the only point of maximum of v .

Let us present some important properties of smooth-peakons, defined in (1.7), which will play a crucial role in the proof of Theorem 3.1. The smooth-peakon ρ_c belongs to $H^3(\mathbb{R}) \hookrightarrow C^2(\mathbb{R})$ (by the Sobolev embedding) since φ_c belongs to $H^1(\mathbb{R})$ (defined in (1.6)). It is a positive even function, which admits a single maximum $c/6$ at point 0, and decays at infinity to 0 (see Fig. 1a). Its derivative ρ'_c belongs to $H^2(\mathbb{R}) \hookrightarrow C^1(\mathbb{R})$, it is an odd function, which vanishes only at the origin and has negative values on $[0, +\infty[$. It admits a single minimum $-c/12$ at point $\ln 2$ and tends at infinity to 0 (see Fig. 1b). Its second derivative ρ''_c belongs to $H^1(\mathbb{R}) \hookrightarrow C^0(\mathbb{R})$, it is an even function, which vanishes at $\pm \ln 2$, takes positive values on $] -\infty, -\ln 2[\cup] \ln 2, +\infty[$ and negative values on $[-\ln 2, \ln 2]$. It admits a single minimum $-c/3$ at point 0 and two maxima $c/24$ at points $\pm \ln 4$, and decays at infinity to 0 (see Fig. 1c).

Next, we will need the following estimates.

Lemma 3.2 (C^0 , C^1 and C^2 Approximations). *Let $u \in L^2(\mathbb{R})$ with $y = (1 - \partial_x^2)u \in \mathcal{M}^+(\mathbb{R})$. If $\|u - \varphi_c\|_{\mathcal{H}} \leq \varepsilon^{1/4}$, then*

$$\|u - \varphi_c\|_{C^0(\mathbb{R})} + \|v - \rho_c\|_{C^2(\mathbb{R})} \leq O(\varepsilon^{1/8}) \quad (3.8)$$

and

$$\|v - \rho_c\|_{C^1(\mathbb{R})} \leq O(\varepsilon^{1/4}). \quad (3.9)$$

Proof. Let us begin with the second estimate. From the definition of $E(\cdot)$ and \mathcal{H} (see respectively (1.2) and (1.4)), one can see that $\|u\|_{\mathcal{H}}$ is equivalent to $\|v\|_{H^2(\mathbb{R})}$, since $\|v\|_{H^2(\mathbb{R})} \leq \|u\|_{\mathcal{H}} \leq 5\|v\|_{H^2(\mathbb{R})}$. Then, assumption u is \mathcal{H} -close to φ_c implies that v is H^2 -close to ρ_c . Now, using the Sobolev embedding of $H^2(\mathbb{R})$ into $C^1(\mathbb{R})$, we deduce (3.9).

For the first estimate, note that the assumption $y = (1 - \partial_x^2)u \geq 0$ implies that $u = (1 - \partial_x^2)^{-1}y \geq 0$ and satisfies $|u_x| \leq u$ on \mathbb{R} (see (2.3)). Then, applying triangular inequality, and using that $|\varphi'_c| = \varphi_c$ on

\mathbb{R} and (2.4), we have

$$\begin{aligned}\|u - \varphi_c\|_{H^1(\mathbb{R})} &\leq \|u\|_{H^1(\mathbb{R})} + \|\varphi_c\|_{H^1(\mathbb{R})} \\ &\leq 2\|u\|_{L^2(\mathbb{R})} + 2\|\varphi_c\|_{L^2(\mathbb{R})} \\ &\leq 2\|u - \varphi_c\|_{L^2(\mathbb{R})} + 4\|\varphi_c\|_{L^2(\mathbb{R})} \\ &\leq O(\varepsilon^{1/4}) + O(1),\end{aligned}$$

where $\|\varphi_c\|_{L^2(\mathbb{R})} = c$. Therefore, applying the Gagliardo-Nirenberg inequality and using that $\|u - \varphi_c\|_{\mathcal{H}} \leq \varepsilon^{1/4}$ (with $\mathcal{H} \simeq L^2$), we obtain

$$\begin{aligned}\|u - \varphi_c\|_{C^0(\mathbb{R})} &\leq \|u - \varphi_c\|_{L^2(\mathbb{R})}^{1/2} \|u - \varphi_c\|_{H^1(\mathbb{R})}^{1/2} \\ &\leq O(\varepsilon^{1/8}) \left(O(\varepsilon^{1/8}) + O(1) \right) \\ &\leq O(\varepsilon^{1/8}).\end{aligned}$$

Finally to estimate the second term of the left-hand side of (3.8), we first notice that the continuity of $(4 - \partial_x^2)^{-1}(\cdot)$ from $H^s(\mathbb{R})$ to $H^{s+2}(\mathbb{R})$ and the above estimates ensure that $\|v - \rho_c\|_{H^3} = O(1)$ and $\|v - \rho_c\|_{H^2} = O(\varepsilon^{1/4})$. These last estimates combined with the Gagliardo-Nirenberg inequality yield the result as above. \square

The following lemma specifies the distance to minimize for stability.

Lemma 3.3 (Quadratic Identity; See [7]). *For any $u \in L^2(\mathbb{R})$ and $\xi \in \mathbb{R}$, it holds*

$$E(u) - E(\varphi_c) = \|u - \varphi_c(\cdot - \xi)\|_{\mathcal{H}}^2 + 4c \left(v(\xi) - \frac{c}{6} \right), \quad (3.10)$$

where $v = (4 - \partial_x^2)^{-1}u$ and $\rho_c(0) = c/6$.

Proof. We follow the idea of Constantin and Strauss with the CH equation (see [2], Lemma 1). We compute

$$\begin{aligned}E(u - \varphi_c(\cdot - \xi)) &= E(u) + E(\varphi_c) - 2 \langle (1 - \partial_x^2)\varphi_c(\cdot - \xi), (4 - \partial_x^2)^{-1}u \rangle_{H^{-1}, H^1} \\ &= E(u) + E(\varphi_c) - 2 \langle (1 - \partial_x^2)\varphi_c(\cdot - \xi), v \rangle_{H^{-1}, H^1},\end{aligned} \quad (3.11)$$

where $\langle \cdot, \cdot \rangle_{H^{-1}, H^1}$ denotes the duality bracket $H^{-1}(\mathbb{R}), H^1(\mathbb{R})$. Now, using the definition of $\varphi'_c(\cdot - \xi)$ and integration by parts, we have

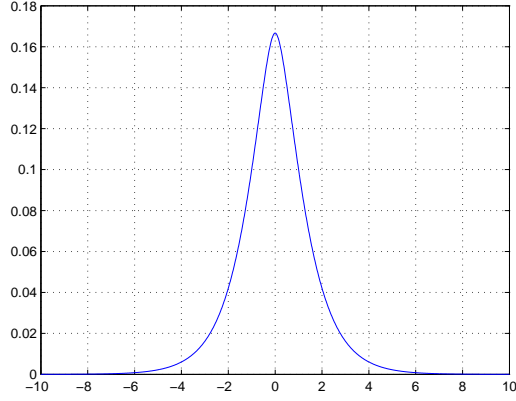
$$\begin{aligned}\langle (1 - \partial_x^2)\varphi_c(\cdot - \xi), v \rangle_{H^{-1}, H^1} &= \int_{\mathbb{R}} v \varphi_c(\cdot - \xi) + \int_{\mathbb{R}} v_x \varphi'_c(\cdot - \xi) \\ &= \int_{\mathbb{R}} v \varphi_c(\cdot - \xi) + \int_{-\infty}^{\xi} v_x \varphi_c(\cdot - \xi) - \int_{\xi}^{+\infty} v_x \varphi_c(\cdot - \xi) \\ &= 2cv(\xi).\end{aligned} \quad (3.12)$$

Recalling that the energy of peakons is given by

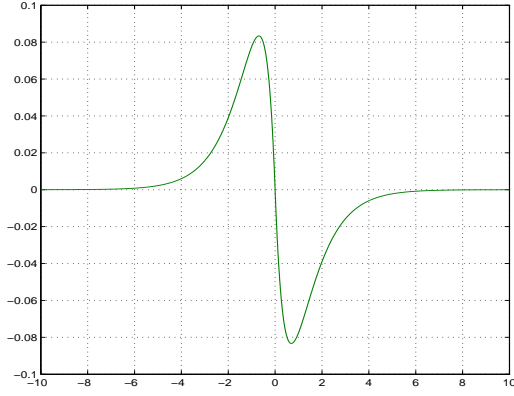
$$\begin{aligned}E(\varphi_c) &= \langle (1 - \partial_x^2)\varphi_c, (4 - \partial_x^2)^{-1}\varphi_c \rangle_{H^{-1}, H^1} = \int_{\mathbb{R}} \rho_c \varphi_c + \int_{\mathbb{R}} \rho'_c \varphi'_c \\ &= \int_{\mathbb{R}} \rho_c \varphi_c + \int_{-\infty}^0 \rho'_c \varphi_c - \int_0^{+\infty} \rho'_c \varphi_c = 2c\rho_c(0) = \frac{c^2}{3},\end{aligned} \quad (3.13)$$

we obtain the lemma. \square

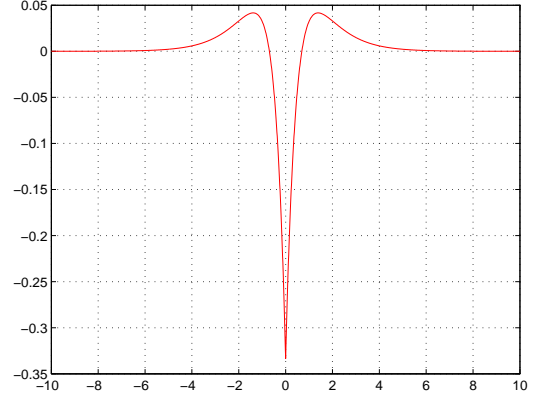
Now we will study carefully the local extrema of $v = (4 - \partial_x^2)^{-1}u$. Let $u \in L^2(\mathbb{R})$ with $y = (1 - \partial_x^2)u \in \mathcal{M}^+(\mathbb{R})$, and assume that (3.6) holds for some $z \in \mathbb{R}$. We consider the interval in which the mass of



(a) $\rho(x) = \frac{1}{3}e^{-|x|} - \frac{1}{6}e^{-2|x|}$ profile.



(b) $\rho'(x) = (\frac{1}{3}e^{-|x|} - \frac{1}{3}e^{-2|x|})_{x<0} + (\frac{1}{3}e^{-2|x|} - \frac{1}{3}e^{-|x|})_{x>0}$ profile.



(c) $\rho''(x) = \frac{1}{3}e^{-|x|} - \frac{2}{3}e^{-2|x|}$ profile

Figure 1: Variation of the smooth-peakon with the amplitude $1/6$ at initial time.

smooth-peakons is concentrated, and the interval in which the mass of second derivative of smooth-peakons is strictly negative. In the sequel of this paper, the notation $\alpha \simeq \beta$ means that $0.9 \times \beta \leq \alpha \leq 1.1 \times \beta$. We set, for any $z \in \mathbb{R}$,

$$\Theta_z = [z - 6.7, z + 6.7], \quad \text{where } 6.7 \simeq \ln \left(\frac{20}{20 - \sqrt{399}} \right), \quad (3.14)$$

and

$$\mathcal{V}_z = [z - \ln\sqrt{2}, z + \ln\sqrt{2}]. \quad (3.15)$$

One can clearly see that \mathcal{V}_0 is a subset of Θ_0 (since $20/(20 - \sqrt{399}) > \sqrt{2}$). We chose the values ± 6.7 such that $\rho_c(\pm 6.7) \simeq c/2400 \simeq 4.1 \times 10^{-4}c$ as in [6]. Also, we have $\rho'_c(-6.7) = -\rho'_c(6.7) \simeq 4.1 \times 10^{-4}c$ and $\rho''_c(\pm 6.7) \simeq 4.1 \times 10^{-4}c$. Then ρ_c , ρ'_c and ρ''_c are very small with respect to the amplitude c on $\mathbb{R} \setminus \Theta_0$.

We claim the following result.

Lemma 3.4 (Uniqueness of the Local Maximum). *Let $u \in L^2(\mathbb{R})$ with $y = (1 - \partial_x^2)u \in \mathcal{M}^+(\mathbb{R})$, that satisfies (3.6) for some $z \in \mathbb{R}$. There exists $\varepsilon_0 > 0$ only depending on the speed c , such that if $0 < \varepsilon < \varepsilon_0$,*

then the function $v = (4 - \partial_x^2)^{-1}u$ admits a unique local extremum on Θ_z . This extremum is a maximum, and it holds

$$v(x) \leq \frac{c}{300}, \quad \forall x \in \mathbb{R} \setminus \Theta_z, \quad (3.16)$$

$$u(x) \leq \frac{c}{300}, \quad \forall x \in \mathbb{R} \setminus \Theta_z. \quad (3.17)$$

Proof. The key is to study the impact of the assumption $y \in \mathcal{M}^+(\mathbb{R})$ on v . First, let us show that $|v_x| \leq 2v$ on \mathbb{R} . We recall that from the assumption $y \geq 0$, we have $u \geq 0$ and $v \geq 0$ on \mathbb{R} . According to the definition of v , we have for all $x \in \mathbb{R}$,

$$v(x) = \frac{e^{-2x}}{4} \int_{-\infty}^x e^{2x'} u(x') dx' + \frac{e^{2x}}{4} \int_x^{+\infty} e^{-2x'} u(x') dx'$$

and

$$v_x(x) = -\frac{e^{-2x}}{2} \int_{-\infty}^x e^{2x'} u(x') dx' + \frac{e^{2x}}{2} \int_x^{+\infty} e^{-2x'} u(x') dx',$$

which yields

$$|v_x(x)| \leq 2v(x), \quad \forall x \in \mathbb{R}. \quad (3.18)$$

Second, let us show that $u \leq 6v$ on \mathbb{R} . Using the Fourier transform, one can check that

$$\begin{aligned} (1 - \partial_x^2)^{-1}(4 - \partial_x^2)^{-1}(\cdot) &= \mathcal{F}^{-1} \left[\frac{1}{3(1 + \omega^2)} - \frac{1}{3(4 + \omega^2)} \right] (\cdot) \\ &= \frac{1}{3}(1 - \partial_x^2)^{-1}(\cdot) - \frac{1}{3}(4 - \partial_x^2)^{-1}(\cdot), \end{aligned} \quad (3.19)$$

and one can rewrite v as

$$v = (4 - \partial_x^2)^{-1}(1 - \partial_x^2)^{-1}y = \frac{1}{3}(1 - \partial_x^2)^{-1}y - \frac{1}{3}(4 - \partial_x^2)^{-1}y. \quad (3.20)$$

Then for all $x \in \mathbb{R}$,

$$\begin{aligned} u(x) - 6v(x) &= -(1 - \partial_x^2)^{-1}y(x) + 2(4 - \partial_x^2)^{-1}y(x) \\ &= -\frac{1}{2} \int_{\mathbb{R}} e^{-|x-x'|} y(x') dx' + \frac{1}{2} \int_{\mathbb{R}} e^{-2|x-x'|} y(x') dx' \\ &\leq 0, \end{aligned} \quad (3.21)$$

since $e^{-2|\cdot|} \leq e^{-|\cdot|}$ on \mathbb{R} .

We are now ready to prove the uniqueness of local maxima in Θ_z . Let us first study the sign of v_{xx} on \mathcal{V}_z . One can easily check that for all $x \in \mathcal{V}_0$,

$$\rho_c''(x) \leq \frac{\sqrt{2}-2}{6}c. \quad (3.22)$$

Then, combining (3.8) and (3.22), taking $0 < \varepsilon < \varepsilon_0$ with $\varepsilon_0 \ll 1$, we have for all $x \in \mathcal{V}_z$,

$$v_{xx}(x) \leq \frac{\sqrt{2}-2}{6}c + O(\varepsilon^{1/4}) \leq \frac{\sqrt{2}-2}{600}c < 0,$$

which implies that v_x is strictly decreasing on \mathcal{V}_z . Let us study the sign of v_x on $\Theta_z \setminus \mathcal{V}_z$. One can easily check that

$$\rho_c'(-\ln\sqrt{2}) = \frac{\sqrt{2}-1}{6}c \quad \text{and} \quad \rho_c'(\ln\sqrt{2}) = -\frac{\sqrt{2}-1}{6}c, \quad (3.23)$$

and that $\rho'_c(x) \geq 10^{-4}c$ for all $x \in [-6.7, -\ln\sqrt{2}]$. Then using (3.9) and taking $0 < \varepsilon < \varepsilon_0$ with $\varepsilon_0 \ll 1$, we have $v_x(x) \geq 4 \times 10^{-5}c > 0$ for all $x \in [z - 6.7, z - \ln\sqrt{2}]$. Proceeding in the same way, we obtain $v_x(x) \leq -4 \times 10^{-5}c < 0$ for all $x \in [z + \ln\sqrt{2}, z + 6.7]$. Since v_x is strictly decreasing on \mathcal{V}_z and changes sign, v_x vanishes once on \mathcal{V}_z and thus on Θ_z . Hence, v admits a single local extremum on Θ_z , which is a maximum since $v_{xx} < 0$ on \mathcal{V}_z .

Now, using that ρ_c is increasing on \mathbb{R}^- , (3.9) and taking $0 < \varepsilon < \varepsilon_0$ with $\varepsilon_0 \ll 1$, it holds for all $x \in]-\infty, z - 6.7[$,

$$v(x) = \rho_c(x - z) + O(\varepsilon^{1/4}) \leq \frac{c}{2400} + O(\varepsilon^{1/4}) \leq \frac{c}{300}.$$

Proceeding in the same way for $x \in]z + 6.7, +\infty[$, we obtain (3.16).

Combining (3.8), (3.21) and proceeding as for the estimate (3.16), we get (3.17). Note that $\varphi_c(\pm 6.7) \simeq 1.2 \times 10^{-3}c$. This completes the proof of the lemma. \square

Under the assumptions of Lemma 3.4, v has got a unique point of global maximum on \mathbb{R} . In the sequel of this section, we will denote by ξ this point of global maximum and we set $M = v(\xi) = \max_{x \in \mathbb{R}} v(x)$. The next two lemmas can be directly deduced from the similar lemmas established in [7] (see also [6]).

Lemma 3.5 (Connection Between $E(\cdot)$ and M^2 ; See [7]). *Let $u \in L^2(\mathbb{R})$ and $v = (4 - \partial_x^2)^{-1}u \in H^2(\mathbb{R})$. Define the function g by*

$$g(x) = \begin{cases} 2v + v_{xx} - 3v_x, & x < \xi, \\ 2v + v_{xx} + 3v_x, & x > \xi. \end{cases} \quad (3.24)$$

Then it holds

$$\int_{\mathbb{R}} g^2(x) dx = E(u) - 12M^2. \quad (3.25)$$

Lemma 3.6 (Connection Between $F(\cdot)$ and M^3 ; See [7]). *Let $u \in L^2(\mathbb{R})$ and $v = (4 - \partial_x^2)^{-1}u \in H^2(\mathbb{R})$. Define the function h by*

$$h(x) = \begin{cases} -v_{xx} - 6v_x + 16v, & x < \xi, \\ -v_{xx} + 6v_x + 16v, & x > \xi. \end{cases} \quad (3.26)$$

Then it holds

$$\int_{\mathbb{R}} h(x) g^2(x) dx = F(u) - 144M^3. \quad (3.27)$$

Sketch of proof. The proof of Lemmas 3.5-3.6 follows by direct computation, using integration by parts, with $v_x(\xi) = 0$ and $v(\pm\infty) = v_x(\pm\infty) = v_{xx}(\pm\infty) = 0$. See [7] (also [6]) to undersand the technique. \square

We can now connect the conservation laws.

Lemma 3.7 (Connection Between $E(\cdot)$ and $F(\cdot)$). *Let $u \in L^2(\mathbb{R})$, with $y = (1 - \partial_x^2)u \in \mathcal{M}^+(\mathbb{R})$, that satisfies (3.6) for some $z \in \mathbb{R}$. There exists $\varepsilon_0 > 0$ only depending on the speed c , such that if $0 < \varepsilon < \varepsilon_0$, then it holds*

$$M^3 - \frac{1}{4}E(u)M + \frac{1}{72}F(u) \leq 0. \quad (3.28)$$

Proof. The key is to show that $h \leq 18M$ on \mathbb{R} . Note that by (3.9) we know that $18M \geq c/4$ and that Lemma 3.4 ensures that $\xi \in \Theta_z$ for ε_0 small enough. Let us set $\lambda = z - 6.7$, $\mu = z + 6.7$, and rewrite the function h as

$$h(x) = \begin{cases} -v_{xx} - 6v_x + 16v, & x < \lambda, \\ u - 6v_x + 12v, & \lambda < x < \xi, \\ u + 6v_x + 12v, & \xi < x < \mu, \\ -v_{xx} + 6v_x + 16v, & x > \mu. \end{cases}$$

If $x \in \mathbb{R} \setminus \Theta_z$, using that $v_{xx} = 4v - u$, (3.16) and (3.17), it holds

$$h \leq |v_{xx}| + 6|v_x| + 16v \leq u + 32v \leq \frac{c}{9} \leq 18M.$$

If $\lambda < x < \xi$, then $v_x \geq 0$, and using that $u \leq 6v$ on \mathbb{R} , we have

$$h = u - 6v_x + 12v \leq 18v.$$

If $\xi < x < \mu$, then $v_x \leq 0$, and similarly using that $u \leq 6v$ on \mathbb{R} , we get

$$h = u + 6v_x + 12v \leq 18v.$$

Therefore, it holds

$$h(x) \leq 18 \max_{x \in \mathbb{R}} v(x) = 18M, \quad \forall x \in \mathbb{R}. \quad (3.29)$$

Now, combining (3.25), (3.27) and (3.29), we get

$$F(u) - 144M^3 = \int_{\mathbb{R}} h(x)g^2(x)dx \leq \|h\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} g^2(x)dx \leq 18M(E(u) - 12M^2),$$

and we obtain the lemma. \square

Proof of Theorem 3.1. We argue as El Dika and Molinet in [4]. As noticed after the statement of the theorem, it suffices to prove (3.7) assuming that $u \in L^2(\mathbb{R})$ satisfies (3.1), (3.2) and (3.6). We recall that $M = v(\xi) = \max_{x \in \mathbb{R}} v(x)$ and we set $\delta = c/6 - M$. We first remark that if $\delta \leq 0$, combining (3.4) and (3.10), it holds

$$\|u - \varphi_c(\cdot - \xi)\|_{\mathcal{H}} \leq |E(u_0) - E(\varphi_c)|^{1/2} \leq O(\varepsilon),$$

that yields the desired result. Now suppose that $\delta > 0$, that is the maximum of the function v is less than the maximum of ρ_c . Combining (3.4), (3.5) and (3.28), we get

$$M^3 - \frac{1}{4}E(\varphi_c)M + \frac{1}{72}F(\varphi_c) \leq O(\varepsilon^2).$$

Using that $E(\varphi_c) = c^2/3$ and $F(\varphi_c) = 2c^3/3$, our inequality becomes

$$\left(M - \frac{c}{6}\right)^2 \left(M + \frac{c}{3}\right) \leq O(\varepsilon^2).$$

Next, substituting M by $c/6 - \delta$ and using that $[M + c/3]^{-1} < 3/c$, we obtain

$$\delta^2 \leq O(\varepsilon^2) \Rightarrow \delta \leq O(\varepsilon). \quad (3.30)$$

Finally, combining (3.4), (3.10) and (3.30), we infer that

$$\|u - \varphi_c(\cdot - \xi)\|_{\mathcal{H}} \leq C\sqrt{\varepsilon},$$

where $C > 0$ only depends on the speed c . This completes the proof of the stability of peakons.

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References

- [1] Adrian Constantin and David Lannes. The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations. *Arch. Ration. Mech. Anal.*, 192(1):165–186, 2009.
- [2] Adrian Constantin and Walter A. Strauss. Stability of peakons. *Comm. Pure Appl. Math.*, 53(5):603–610, 2000.
- [3] A. Degasperis, D. D. Kholm, and A. N. I. Khon. A new integrable equation with peakon solutions. *Teoret. Mat. Fiz.*, 133(2):170–183, 2002.
- [4] Khaled El Dika and Luc Molinet. Stability of multipeakons. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(4):1517–1532, 2009.
- [5] Joachim Escher, Yue Liu, and Zhaoyang Yin. Global weak solutions and blow-up structure for the Degasperis-Procesi equation. *J. Funct. Anal.*, 241(2):457–485, 2006.
- [6] André Kabakouala. Stability in the energy space of the sum of N peakons for the Degasperis-Procesi equation. *J. Differential Equations*, 259(5):1841–1897, 2015.
- [7] Zhiwu Lin and Yue Liu. Stability of peakons for the Degasperis-Procesi equation. *Comm. Pure Appl. Math.*, 62(1):125–146, 2009.
- [8] Yue Liu and Zhaoyang Yin. Global existence and blow-up phenomena for the Degasperis-Procesi equation. *Comm. Math. Phys.*, 267(3):801–820, 2006.